

# $N = 2$ Generalized Superconformal Quiver Gauge Theory

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## Abstract

Four dimensional  $N = 2$  generalized superconformal field theory can be defined by compactifying six dimensional  $(0, 2)$  theory on a Riemann surface with regular punctures. In previous studies, gauge coupling spaces of those theories are identified with the moduli space of Riemann surface with marked points  $M_{g,n}$ . We show that the weakly coupled gauge group description corresponds to a stable nodal curve and different duality frames correspond to different stable nodal curves of the same punctured Riemann surface, so the coupling space is indeed Deligne-Mumford compactification  $\bar{M}_{g,n}$ . We also give an algorithm to determine the weakly coupled gauge group and the new appearing puncture if the gauge group is completely decoupled, similarly we can also determine the matter ending only on this gauge group. The same nodal curve can be used to represent the conformal block of two dimensional conformal field theory, therefore we establish an one to one correspondence between the conformal block of 2d theory in a certain channel and the weakly coupled 4d theory. The information about the gauge group tells us what is the intermediate state in the conformal block. We can also determine whether the correlation function of 2d theory can be written as the product of three point functions and the conformal block using gauge theory.

# 1 Introduction

Four dimensional  $N = 2$  superconformal  $A$  type quiver gauge theory (the quiver has the shape of  $A_n$  type dynkin diagram) is realized as the six dimensional  $(0, 2)$  SCFT compactified on a punctured Riemann surface [1, 2]. The gauge coupling constants of four dimensional theory are identified with the complex structure moduli of the punctured Riemann surface  $M_{g,n}$ . The  $N = 2$  dualities are interpreted geometrically as the conformal mapping group of the Riemann surface. It is shown in [3, 4] that the BPS equation governing the compactification is the Hitchin equation [5, 6] defined on the Riemann surface; At the puncture, the solution of the Hitchin equation has a simple singularity. The moduli space of solutions to Hitchin's equation is a Hyperkahler manifold. In one of the complex structures, there is Hitchin's fibration and we can write the corresponding spectral curve which is identified with the Seiberg-Witten fibration [7, 8], so the IR behavior of the four dimensional theory can be determined by the Hitchin's equation.

We can engineer a large class of  $N = 2$  superconformal field theories (SCFT) by putting a collection of simple singularities on the Riemann surface. We may call the corresponding gauge theory as the generalized quiver gauge theory. In general, these theories have no lagrangian description. However, It would be interesting to determine the weakly coupled gauge groups in this quiver. For the  $A$  type linear quiver, Gaiotto [2] argued that the weakly coupled gauge group description corresponds to the degeneration limit of the Riemann surface, different weakly coupled descriptions correspond to different degeneration limits of the Riemann surface. This can be easily applied to the generalized quiver gauge theory. The difficulty is to determine the weakly coupled gauge group in the general case. We will solve this problem in this paper and give an algorithm to calculate the weakly coupled gauge groups in any duality frame based on the information encoded in the puncture.

For  $A$  type quiver, the six dimensional construction involves Riemann sphere with two generic punctures and several simple punctures. In one duality frame, both the gauge group and the matter fields can be shown to be weakly coupled, and this is just the  $A$  type quiver we start with; In other weakly coupled gauge group descriptions, the matter fields may not be weakly coupled, see the original example by Argyres-Seiberg [1]. For the generalized quiver, it may be impossible to find any duality frame in which all the gauge groups and matter fields are weakly coupled, and we do not have the conventional lagrangian description. This may not be a problem since we can define the theory as a two dimensional conformal field theory [9–11]. There is no difficulty to study the correlation function with several insertion of primary states, however, it is not easy to determine the intermediate state from the conformal field theory, the gauge theory calculation will tell us what the intermediate state is. Another motivation of studying the generalized quiver gauge theory is that these theories have interesting three dimensional mirror pair [12] if we compactify the theory down to three dimension [13].

We found that the weakly coupled description is in one to one correspondence with the stable nodal curve of the punctured Riemann surface. The stable nodal curve is

certain kind of singular limit of the punctured Riemann surface and it represents the boundary point of the moduli space of Riemann surface, therefore the coupling space is identified with Deligne-Mumford compactification of moduli space  $\bar{M}_{g,n}$  [14]. It is interesting to note that the same space plays an important role in 2d conformal field theory [15]. In that context, different nodal curves correspond to different channels of the correlation functions, so we establish an one to one correspondence between the weakly coupled 4d theory in one duality frame and 2d correlation function in one channel. The S duality is related to crossing symmetry of two dimensional CFT. When the gauge group is completely decoupled, the original gauge theory decomposes into three parts, a decoupled gauge group and possibly fundamental fields on it, and two generalized quivers (sometimes, they are not generalized quiver as we discuss in the main part of this paper). The original Riemann surface splits into two Riemann surfaces, and there is one new appearing puncture on each Riemann surface. We then give an algorithm to calculate the decoupled gauge group and the new puncture by matching the Coulomb branch moduli. Similarly, by matching the Higgs branch moduli, we can find the matter ending on a single gauge group.

It is well-known that two dimensional conformal field theory can be sewed from three punctured spheres [16]. In the complete degeneration limit of the gauge theory, the Riemann surface degenerates into a collection of three punctured sphere. Some of the three punctured spheres represent the matter system like an bi-fundamental fields or strongly coupled isolated SCFT, the sewing process in gauge theory term is to gauge the diagonal flavor symmetry of two three punctured sphere. So we may wonder whether any four dimensional  $N = 2$  theory can be thought of as gauging the flavor symmetry of the matter system represented by three punctured sphere. Generically this is not true since not every three punctured sphere in the complete degeneration limit represents a matter system. With this understanding, we then give a conjecture whether the corresponding two dimensional conformal field theory correlation function can be written as the product of three point function and conformal block.

This paper is organized as follows: In section II, we review six dimensional description of a large class of four dimensional  $N = 2$  SCFT; In section III, we introduce the definition of stable nodal curve and argue that the weakly coupled description corresponds to nodal curve; In section IV, we give an algorithm to identify the weakly coupled gauge group; In section V, we discuss some examples by applying the algorithm developed in section IV; In section VI, we give an interpretation about the matter system which is represented by three punctured sphere. Finally, we give a discussion about the future research direction.

## 2 Review

A large class of four dimensional  $N = 2$  superconformal gauge theories can be engineered as the six dimensional  $(0, 2)$   $A_{N-1}$  SCFT compactified on a Riemann surface with or without marked points. One can argue that the BPS equation for the com-

pactification is Hitchin's equation. At the marked points, the solution to Hitchin's equation has the simple singularity. The Hitchin equation can also be defined for  $N = 2$  asymptotical free theory [17], in this case, we need to turn on irregular singularity at the puncture. In this paper, we only study the SCFT case and only consider the regular singularity to Hitchin's equation.

Let's pick  $SU(N)$  gauge group and write  $g$  for its lie algebra,  $t$  the lie algebra of the maximal torus  $T$ . Hitchin's equation is

$$\begin{aligned} F - \phi \wedge \phi &= 0 \\ D\phi &= D * \phi = 0, \end{aligned} \tag{1}$$

where  $F$  is the curvature of the connection  $A_\mu$  of a vector bundle defined on Riemann surface  $\Sigma$ ,  $\phi$  is the one form called Higgs field. The local behavior of conformal invariant solution to Hitchin's equation with regular singularity is [18](we consider one singularity here, multiple singularities can be studied similarly)

$$\begin{aligned} A &= \alpha d\theta + \dots \\ \phi &= \beta \frac{dr}{r} - \gamma d\theta + \dots \end{aligned} \tag{2}$$

where  $\alpha, \beta, \gamma \in t$  (more precisely,  $\alpha$  takes value in maximal torus  $T$ ) and less singular terms are not written explicitly;  $z = re^{i\theta}$  is the local holomorphic coordinate. The moduli space of Hitchin's equation with the above behavior around the singularity is denoted as  $M_H(\Sigma, \alpha, \beta, \gamma)$ .  $M_H(\Sigma, \alpha, \beta, \gamma)$  is a hyperkahler manifold and has a family of complex structures parameterized by  $CP^1$ . These complex structures generically depend on the complex structure of the Riemann surface with some exception which will be important to us later. In one distinguished complex structure I, each point on moduli space represents a Higgs bundle on Riemann surface; the complex structure modulus is  $\beta + i\gamma$ , and the kahler modulus is  $\alpha$ . In the same complex structure, there is Hitchin's fibration which can be identified as the Seiberg-Witten fibration. In the study of Seiberg-Witten curve of four dimensional theory, only complex structure of the moduli space matters. since the residue of the Higgs field is  $\sigma = \frac{1}{2}(\beta + i\gamma)$  which determines the complex structure of the moduli space, we will focus on the coefficient of Higgs field.

Assume  $\alpha$  is regular with a Levi subgroup  $L$ , then the residue of Higgs field is taking value in the lie algebra of the parabolic group determined by  $\alpha$  modulo an element of the lie algebra  $\mathfrak{n}$  of the unipotent radical of the parabolic group. The massless theory corresponds to  $\beta, \gamma \rightarrow 0$ , in this case, the above solution is not trivial, less singular terms can be added, and one can show that the Higgs field has a simple pole with residue in  $\mathfrak{n}$ . Interestingly, the nilpotent element of  $sl_N$  is classified by Yang tableaux with total boxes  $N$ , so we can label the singularity by the Yang tableaux, this is in agreement with the result by studying Seiberg-Witten curve of  $N = 2$  SCFT [2]. One can also show that Hitchin's fibration is the same as the Seiberg-Witten fibration [4]. The massive theory is derived by deforming to nonzero  $\beta, \gamma$  regular with  $L$ , the form of  $\beta + i\gamma$  is also determined by the same Yang tableaux.

The local moduli space of solution is described by the adjoint orbit  $\mathcal{O}_i$  of the complex lie algebra  $sl(N, c)$  (the relation of the adjoint orbit to moduli space of Nahm's equation can be found in [19, 20]). The nilpotent orbit is used to describe the massless theory while the semi-simple orbit is used to describe the mass-deformed theory. The nilpotent orbit is classified by the Young tableaux  $[n_1, n_2, \dots, n_r]$  (for an introduction to nilpotent orbit, see [21]), and the mass-deformed theory can be also read from Young tableaux: there are a total of  $n_1$  mass parameters and the degeneracy of each mass parameter is equal to the number of boxes on each column. There are only  $n_1 - 1$  independent mass parameters because of the traceless condition. The dimension of the local moduli space is equal to the dimension of the orbit  $\mathcal{O}_i$ :

$$\dim(\mathcal{O}_i) = N^2 - \sum r_j^2, \quad (3)$$

where  $r_j$  is the height of  $j$ th column of the Young tableaux. We can also read the flavor symmetry from the Yang tableaux, it is

$$S(\prod_{l_h > 0} U(l_h)), \quad (4)$$

Where  $l_h$  is the number of columns with height  $h$ , the maximal possible simple subgroup is  $SU(n_1)$ . The contribution of this puncture to the Higgs branch moduli is (see the derivation in appendix A)

$$l_i = \frac{1}{2} \left( \sum_{k=1}^r n_k^2 - N \right). \quad (5)$$

The simple puncture contributes 1 and the full puncture contribute  $\frac{1}{2}(N^2 - N)$ .

The Hitchin's moduli space on the sphere can be modeled as the quotient

$$(\mathcal{O}_1 \times \mathcal{O}_2 \dots \times \mathcal{O}_m) / G, \quad (6)$$

where  $G$  is the complex gauge group, and the total dimension is the sum of the local dimension minus the dimension of the gauge group. The minimal nilpotent orbit has partition  $[2, 1, \dots, 1]$  and the mass-deformed theory has only one mass parameter, we call this kind of singularity as the simple singularity, the local dimension of moduli space is  $2N - 2$  using (3). The maximal nilpotent orbit has partition  $[N]$ , and its dimension is  $N^2 - N$ , we call it full singularity. The total dimension of the Coulomb branch is (half of the dimension of Hitchin's moduli space)

$$\frac{1}{2} \sum \dim(\mathcal{O}_i) + (g - 1)(N^2 - 1). \quad (7)$$

Similarly, the total dimension of the Higgs branch is (In fact, for higher genus case, the number is not really the dimension of the Higgs branch, since it might be negative; This number is counting the dimension of the matter minus that of gauge groups.)

$$\sum_i l_i + (1 - g)(N - 1). \quad (8)$$

The above description tells us what is the contribution of a single puncture to the Coulomb branch, we also want to know what is the total dimension of degree  $i$  operators in Coulomb branch. This can be seen from Seiberg-Witten curve. The Seiberg-Witten curve is the spectral curve

$$\det(x - \Phi(z)) = x^N - \sum_{i=2}^N \phi_i(z) x^{N-i} = 0, \quad (9)$$

where  $\Phi(z)$  is the holomorphic part of the Higgs field and  $\phi_i$  is the degree  $i$  meromorphic differential on the Riemann surface parameterized by  $z$ . For the massless theory, the pole of order of  $\phi_i$  at  $j$ th puncture is

$$p_i^{(j)} = i - s_i^{(j)}, \quad (10)$$

where  $s_i^{(j)}$  is the height of  $i$ th box in the Young tableaux for the  $j$  the puncture. The coefficient of this differential represents dimension  $i$  operator parameterizing the Coulomb branch of quiver gauge theory, the total dimension of this differential is

$$d_i = \sum_{j=1}^n p_i^{(j)} - 2i + 1. \quad (11)$$

A large class of four dimensional  $N = 2$  superconformal field theories can be constructed by putting together different punctures on Riemann surface, most of them do not have the conventional lagrangian description. It is interesting to compare the  $UV$  parameters of the gauge theory with known lagrangian description and the parameters needed to define Hitchin's moduli space. Let's consider  $A$  type quiver gauge theory with gauge group  $\sum_{i=1}^n SU(k_i)$  with  $k_1 < k_2 < \dots < k_r = \dots = k_s > k_{s+1} > \dots > k_n$  with  $k_r = \dots = k_s = N$ . The matter contents are the bi-fundamental hypermultiplet between the adjacent gauge groups and the fundamentals  $d_\alpha$  on each node to make gauge theory conformal, indeed

$$d_\alpha = 2k_\alpha - k_{\alpha-1} - k_{\alpha+1}. \quad (12)$$

Hitchin's system involves  $n+1$  simple regular singularities and two generic regular singularities on Riemann sphere. The two generic regular singularities are used to describe two quiver tails, we study left quiver tail  $SU(k_1) - SU(k_2) - \dots - SU(k_r)$  and right quiver can be treated similarly. It is associated with a Young tableaux with partition  $[n_1, \dots, n_r]$ , where  $n_\alpha = k_\alpha - k_{\alpha-1}$ , there are  $n_1 - 1$  mass parameters from this singularity. We also have  $\sum_{\alpha=1}^r d_\alpha = k_1 = n_1$ .

Let's compare the  $UV$  parameters of the gauge theory with the parameters for the Hitchin system. The  $UV$  parameters of the gauge theory are the dimensionless gauge couplings, mass parameters for the bi-fundamental and fundamental hypermultiplets. The parameters for the Hitchin system are the complex structure of the Riemann surface and the local parameters around the regular singularities. There are  $n$  dimensionless gauge coupling constants, and these are represented by the complex structure of punctured Riemann sphere, since there are  $n + 3$  punctures on the

sphere and the Riemann surface has  $n$  complex structure moduli. There are  $n - 1$  mass parameters for bi-fundamental hypermultiplets and they are encoded in the parameters of  $n - 1$  regular simple singularities; For the left quiver tail, there are a total of  $n_1$  fundamental fields and  $n_1$  mass parameters, these are described by a generic singularity and a regular simple singularity: the generic regular singularity has  $n_1 - 1$  mass parameters and the simple regular singularity has one. The same analysis applies to the right quiver, so all the parameters are nicely encoded in the Hitchin's system. For the IR behavior, the Seiberg-Witten fibration is identified with Hitchin's fibration, one can check that the dimension of the base of Hitchin's fibration is the same as the dimension of Coulomb branch of gauge theory.

Finally, we want to stress that the UV parameters enter into the Hitchin's system in different ways. The Hitchin's moduli space is a hyperkahler space which is described by fixing the coefficient of the simple pole, these coefficients describes the complex structure and kahler structure of Hitchin's moduli space, and these parameters are identified with the mass parameters of the gauge theory, so the hyperkahler structure depends on the mass parameters. On the other hand, the UV gauge coupling constants are identified with the complex structure of the Riemann surface, but some of the complex structures of the Hitchin's moduli space does not depend on the complex structure of Riemann surface, so the hyperkahler structure is independent of the gauge coupling constant.

As we discussed in the introduction, it seems that the Riemann surface encodes all the information about the gauge theory; On the other hand, two dimensional conformal field theory is naturally defined on the Riemann surface with punctures. AGT [9] found the surprising relation between the Nekrasov partition function and conformal blocks of Liouville theory; The relation is extended to asymptotical free cases and  $SU(N)$  conformal theory [10, 11]. These relations have a lot of extensions and went through a lot of checks [22–46]. One can also compare the expectation values of Wilson-t'hooft loops and surface operators with correlation function of 2d CFT [47–56]. AGT relation can be understood from matrix model [57] and there are also a lot of developments along this line [58–64]. AGT is explained by using  $\Omega$  deformation and branes [65] and M theory is also useful in understanding the AGT relation [66, 67]. See also the development in understanding gauge theory side [68–74]. The interesting relation between gauge theory and the integrable system is extensively discussed in [75–78].

### 3 The Shape of Generalized Quiver from nodal curve

In the previous section, we argue that gauge coupling constants of four dimensional  $N = 2$  SCFT are identified as the complex structure moduli of a Riemann surface with punctures. In this section, we show that we can determine the structure of quiver with weakly coupled gauge groups from studying the compactification of moduli space.

Consider a two dimensional topological surface  $\Sigma$  with  $g$  handles and  $n$  marked points. This manifold can be made into a complex manifold by defining a complex structure  $J$  on it. A complex structure  $J$  is a local linear map on the tangent bundle that satisfies  $J^2 = -1$  and the integrability condition. Two complex structures are considered equivalent if they are related by a diffeomorphism. The moduli space  $M_{g,n}$  is the space of all the inequivalent complex structure on the surface. By Riemann-Roch this is a space of complex dimension

$$\dim M_{g,n} = 3g - 3 + n. \quad (13)$$

$M_{g,n}$  is a noncompact complex space with singularities. It arises as the quotient of a covering space known as Teichmüller space  $T_{g,n}$ , by a discrete group, conformal mapping class group  $MC_{g,n}$ :

$$M_{g,n} = \frac{T_{g,n}}{MC_{g,n}}. \quad (14)$$

This action typically has fixed points, and the moduli space has orbifold singularities.

There is another useful way to think about the complex structure on  $\Sigma$ . We can think of the point on the moduli space as the conformal class of a metric  $g_{\mu\nu}$ . Indeed, a metric defined a complex structure through

$$J_\mu^\nu = \sqrt{h} \epsilon_{\mu\lambda} h^{\lambda\nu}, \quad (15)$$

with  $\epsilon_{\mu\nu}$  the Levi-Civita symbol. The definition of the complex structure does not depend on the local rescaling of the metric  $g_{\mu\nu}$ , so we can think of the moduli space as the spaces of metric modulo local rescalings and diffeomorphisms.

The moduli space  $M_{g,n}$  is noncompact and has a boundary. The boundary points can be intuitively represented as degenerate surfaces. The degeneration can be thought in two ways; the surface can either form a node-or equivalently a long neck-or two marked points can collide. The process in which two points  $x_1$  and  $x_2$  collide if  $q = x_1 - x_2$  tends to zero can alternatively be described as a process in which a sphere, that contains  $x_1$  and  $x_2$  at fixed distance, pinches off the surface by forming a neck of length  $\log q$ . So the degeneration limit can be thought of the nodal curve. The boundary points can be thought of as in the infinity and we would like to compactify this space. The Deligne-Mumford compactification of  $M_{g,n}$  is achieved by adding some points which represent stable nodal curves.

In the following, we will introduce some basic concepts about the nodal curve. Singular objects play an important role in algebraic geometry. The simplest singularity a complex curve can have is a node. A nodal point of a curve is a point that can be described locally by the equation  $xy = 0$  in  $C^2$ . An example is shown in figure 1a).

We also find the following description of nodal curve very useful. On a surface with node, the node separates the surface into two components, on the neighborhood of each node, we can choose local coordinate disks  $\{z_i : |z_i| < 1\}, i = 1, 2$ . The two disks are glued together at the origin  $z_1, z_2 = 0$  to form the node. We can open the node by introducing one of complex coordinate  $q$  of the moduli space  $M_{g,n}$ .



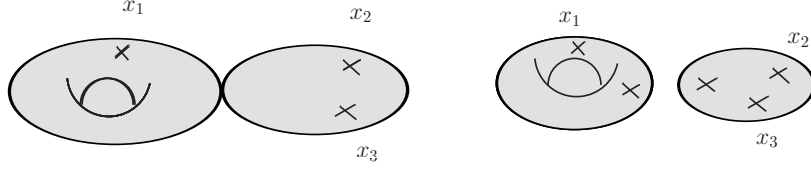


Figure 1: a) A nodal curve; b) The normalization of a.

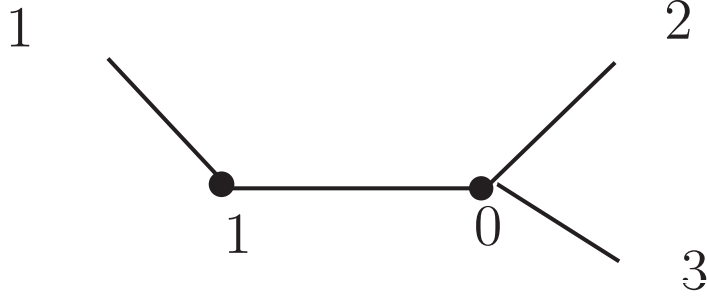


Figure 2: A dual graph for the nodal curve in figure 1

Remove the sub-disks  $|z_i| < |q|^{\frac{1}{2}}$  and attach the resulting pair of annuli at their inner boundaries  $|z_i| = |q|^{\frac{1}{2}}$  by identifying  $z_2 = q/z_1$ . This coordinate neighborhood on the surface is mapped to a single annulus  $|q|^{\frac{1}{2}} < |z| < |q|^{\frac{-1}{2}}$ , by

$$\begin{aligned} z &= q^{1/2}/z_2, \quad \text{if } |q|^{1/2} < |z| \leq 1 \\ z &= q^{1/2}/z_2, \quad \text{if } 1 \leq |z| < |q|^{-1/2}. \end{aligned} \quad (16)$$

As  $q = 0$ , we recover the node. A further transformation  $\omega = (2\pi i)^{-1} \ln z$  pictures the opened node as a long tube. Writing  $q = e^{2\pi i \tau}$ , the length and width is determined by  $\tau$ . The node corresponds to a tube of infinite length. In this description, we see that the moduli is localized on the long tube, and since we identify the moduli with the gauge coupling constant, we can think that the gauge group is represented by the long tube.

We define the normalization of the nodal curve as unglueing its nodes, and add a marked points to each of the components on which the nodes belong to. See Figure 1b) for an example. Each component  $\Sigma_i$  after the normalization is an irreducible component of  $\Sigma$ .

There is another convenient way of describing the nodal curve by drawing a dual graph. The vertices of the dual graph of  $\Sigma$  corresponds to components of  $\Sigma$  (and are labeled by their genus), and the edge correspond to node, we use labeled tails to represent the marked points. An example is shown in Figure 2.

A stable nodal curve is a connected nodal curve such that

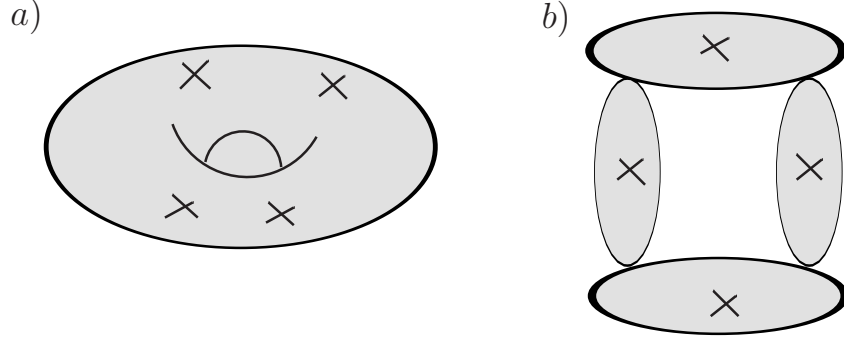


Figure 3: a) A torus with four marked points; b) An irreducible nodal curve of a.

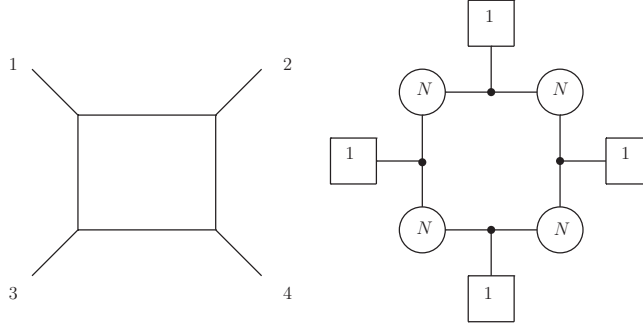


Figure 4: a) The dual graph for the irreducible nodal curve of Figure 3b), we omit the genus 0 on each vertex for simplicity, since for irreducible nodal curve, all components have genus zero; b) Four dimensional gauge theory, we put a gauge group on the internal line, external lines represent the  $U(1)$  flavor symmetries.

(i) Every irreducible component of geometric genus 0 has at least three special points (including the marked points and the nodal points after the normalization).

ii) Every irreducible component of geometric genus 1 has at least one special point.

Deligne-Mumford compactification  $\bar{M}_{g,n}$  includes the points corresponding to the stable curve to moduli space  $M_{g,n}$ .

Let's define an irreducible nodal curve as a curve whose irreducible components are all genus 0 curve with three special points. See Figure 3 for an example, The dual graph for this particular nodal curve is depicted in Figure 4.

Let's consider another genus one example, in this case, the two nodes belong to the same irreducible component after normalization. The degeneration limit and the dual graph are shown in Figure 5.

It is time now to connect the nodal curve to the weakly coupled four dimensional  $N = 2$  quiver we are studying in last section. As we reviewed in last section, each puncture is associated with certain flavor symmetry, and the node or the long neck is

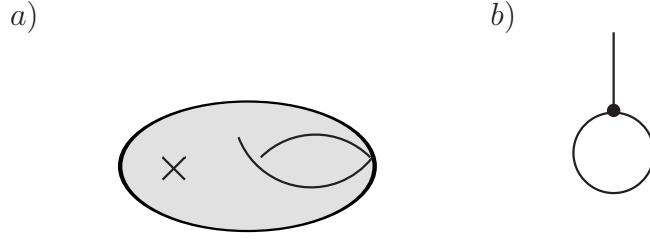


Figure 5: a) Degeneration limit of torus with one marked point; b) The dual graph of a), there is only one irreducible component and the nodes are in the same component so the nodes is represented by a loop connecting to the same vertex.

identified with the weakly coupled gauge group, we have the following identification:

A generalized quiver with weakly coupled gauge group associates with the stable nodal curve and the quiver with all gauge group weakly coupled is the irreducible nodal curve.

In fact, we can read the quiver from the dual graph in figure 4; Let's assume that we compactify  $A_{N-1}$  theory on the Riemann surface, and the punctures are simple punctures. We consider the weakly coupled gauge theory corresponding to irreducible nodal curve. The line ending on only one vertex is the original puncture and represent the flavor symmetry, we call them external line; The line between the nodes represent the gauge groups, we call them internal line. There are three lines connecting each node. In this particular example, for each node, there are two internal lines connecting it and one external line representing a  $U(1)$  flavor symmetry, the gauge theory interpretation is that the two gauge groups connecting to a single node are adjacent and there are bi-fundamental fields connecting them, and the quiver is of the form in Figure 5. In general, we can read the shape of the generalized quiver in any duality frame from the dual graph of the corresponding nodal curve, the difference is that the matter fields between the adjacent gauge group is not necessarily bi-fundamental fields, it maybe a strongly coupled isolated superconformal field theory like  $E_6$  SCFT.

Since for an irreducible nodal curve, each irreducible component is a genus 0 Riemann sphere with three punctures, we may think that each three punctured represents a matter either conventional bi-fundamental fields or strongly coupled isolated SCFT matter. The whole generalized quiver is derived by gluing the matter fields. The gluing process is to gauge the diagonal flavor symmetry of two punctures on two different irreducible components or one component, in the latter case, we add a handle to the Riemann surface. This is true for generalized quiver gauge theory defined by six dimensional  $A_1$  theory with any number of punctures; For  $A_N$  theory, in certain duality frame (for certain irreducible nodal curve) this is true, but generically this is not the case, since some of the three punctured sphere does not represent a matter, we will discuss this more in later sections.

Let's summarize what we learn about the relation between the stable nodal curve

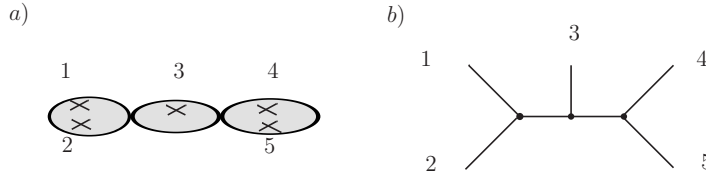


Figure 6: a) A nodal curve with 5 marked points and two vertices. b) The dual graph which resembles the conformal field theory conformal block with 5 external states.

and the four dimensional  $N = 2$  weakly coupled quiver gauge theory. For each stable nodal curve, there is a four dimensional gauge theory for which one or more than one gauge groups become weakly coupled, and the gauge couplings are taking value at the boundary of the moduli space  $M_{g,n}$ . The four dimensional quiver for which all the gauge groups are weakly coupled corresponds to the irreducible nodal curve. In another words, the four dimensional  $N = 2$  SCFT gauge coupling space is  $\bar{M}_{g,n}$ .

It is illuminating to note that  $\bar{M}_{g,n}$  also plays an important role in 2d conformal field theory. Let's consider a two dimensional conformal field theory defined on a Riemann surface. We usually want to calculate the correlation functions with several insertions on the Riemann surface, and the correlation function can be calculated in different channels, say  $s, t$  channels. Recently, AGT [9] found an interesting relation between the partition function of  $N = 2$   $SU(2)$  four dimensional gauge theory and the correlation function of the Liouville theory based on the six dimensional realization. The primary fields at the insertion can be read from the information of the puncture used to describe the gauge theory. It is interesting to note that the different channels for two dimensional correlation function are also in one-to-one correspondence with the nodal curve. The different channels are exactly represented by the dual graph of the nodal curve, the external states are represented as the external line while the intermediate states are represented by the internal line. The intermediate states can be determined by using familiar operator product expansion technics based on the fixed external states. See Figure 6 for an example.

Using the nodal curve, we now establish a one-to-one correspondence between four dimensional gauge theory and two dimensional CFT correlation function. The external lines on dual graph determine the flavor symmetry of the gauge theory and the external states of the two dimensional correlation function, the classification of punctures is given in last section and this leads to the determination of physical states in 2d CFT. The internal lines represent gauge group of gauge theory and the intermediate state of the 2d CFT. We need to determine the gauge group and then we can identify the intermediate state in 2d CFT.

## 4 Determine the Gauge Group

In last section, we established a relation between  $N = 2$  weakly coupled SCFT and the nodal curve. We have shown that the structure of weakly coupled quiver is in a one to one correspondence with the stable node curve. The remaining task is to determine what is the weakly coupled gauge group. To achieve this, it is not enough to have just the gauge group and we need more structure on Riemann surface. Actually, there is a special structure to describe four dimensional theory: there is a Hitchin system defined on the punctured Riemann surface. The base of Hitchin's fibration is identified with the Coulomb branch of four dimensional theory and the dimension of the base is equal to the dimension of the coulomb branch. At each puncture, the local moduli space is modeled by a nilpotent orbit which is labeled by a Yang Tableaux. The contribution of this puncture to total dimension of coulomb branch is given in formula (3); one can even determine the contribution of this puncture to degree  $i$  operators from the Young tableaux (10). We determine the weakly coupled gauge group by matching the coulomb branch moduli of the degeneration limit and the original theory.

Let's first consider degeneration limit of punctured Riemann sphere. The degeneration limit as we described in last section is just the nodal curve. There is a long tube which corresponds to a weakly coupled gauge group of four dimensional gauge theory; When the long tube is completely pinched off, the original Riemann surface decomposes into two parts: a decoupled gauge group and possibly matter ending only on it and two other Riemann spheres. In four dimensional gauge theory terms, the original quiver decomposes into three parts, a decoupled gauge group and the matter only ending on it, and two other quivers. the two new Riemann sphere can be either a quiver gauge theory or a matter system in the case of three punctured sphere, it is even possible that the left part does not have clear physical meaning. We can find the decoupled gauge group by matching the Coulomb branch moduli of the original quiver and the three parts after the decoupling. We make an assumption that the gauge group is a simple gauge group with the form  $SU(k)$ ,  $k \leq N$  or  $USp(2k)$ ,  $k \leq [\frac{N}{2}]$ . This assumption is confirmed in any known example.

To begin with, we give a definition about the irreducible rank  $N$  theory defined on a punctured Riemann sphere. For a four dimensional theory derived from a six dimensional  $A_{N-1}$   $(0, 2)$  theory compactified on a punctured sphere, the Seiberg-Witten curve is

$$x^N - \phi_i x^{N-i} = 0. \quad (17)$$

Here  $\phi_i dz^i$  is a degree  $i$  meromorphic differential defined on Riemann sphere, the dimension  $d_i$  of this differential is

$$d_i = \sum_j p_i^{(j)} - 2i + 1, \quad (18)$$

here  $p_i^j$  is the order of pole at the  $j$ th puncture.  $p_i^{(j)}$  can be read from the Young Tableaux:

$$p_i^{(j)} = i - s_i, \quad (19)$$

where  $s_i$  is the height of  $i$ th box in the Young tableaux.

Now consider the degree  $N$  differential, if the number  $d_N \leq 0$ , which means the number of parameters for  $\phi_N$  is zero, then the Seiberg-Witten curve becomes

$$x(x^{N-1} - \phi_i x^{N-1-i}) = 0, \quad (20)$$

so actually this theory is a rank  $(N-1)$  theory if  $d_{N-1} > 0$ . We call a theory defined by  $A_{N-1}$  compactified on a punctured Riemann surface irreducible if  $d_N > 0$ .

In fact, we only need to determine the decoupled gauge group when two punctures are colliding; We learn from last section that there is a new puncture appearing on irreducible components besides the original marked point. We assume that these points have the same nature as the marked points, i.e. the local behavior of Hitchin's system takes the same form as the marked points, this is justified by studying the conventional four dimensional  $A$  type quiver. If we need to know what happens when more than two punctures collide, we can first collide two punctures, and then consider the Riemann surface with  $n-1$  puncture: a new appearing puncture and  $N-2$  original punctures. We collide another puncture with the new puncture, following this procedure, we can determine any weakly coupled description as long as we know what happened when two generic punctures are colliding.

We consider an irreducible rank  $N$  theory derived from a Riemann sphere with  $n$  punctures. When two punctures are colliding, we are left with a three punctured sphere which we called part 1 and a  $n-1$  punctured part and we call it part 2. The new punctures on two different parts are the same and there is an important relation between the decoupled gauge group  $SU(k)$  or  $USp(k)$  and the new appearing puncture. There is certain flavor symmetry associated with the new appearing gauge group and it must contain a  $SU(k)$  factor, the original quiver is derived by gauging this subgroup of the flavor symmetry. There may be some fundamental fields on the decoupled gauge group so that the gauge group is conformal.

We need to determine what is the Yang Tableaux associated with the new puncture to match the Coulomb branch moduli. This is equivalent to know what is the order of pole of degree  $i$  differential at this new puncture. The method we are using is to match the number of moduli with the original quiver. Consider the degree  $i$  moduli, assume the two colliding punctures contribute to  $\delta_{1i} = p_i^{(1)} + p_i^{(2)}$ , and the other  $n-2$  punctures contribute to  $\delta_{2i}$ . Let's first assume that both components have zero or non-zero degree  $i$  moduli, then we have following two options:

$$(\delta_{1i} + p_i - 2i + 1) + (\delta_{2i} + p_i - 2i + 1) = \delta_{1i} + \delta_{2i} - 2i + 1 \quad (21)$$

or

$$(\delta_{1i} + p_i - 2i + 1) + (\delta_{2i} + p_i - 2i + 1) + 1 = \delta_{1i} + \delta_{2i} - 2i + 1 \quad (22)$$

where  $p_i$  is the contribution from the new puncture to the  $i$ th degree moduli. For the first option, we assume that the decoupled gauge group does not have a degree  $i$  operator. For the second option, we assume that the decoupled gauge group carry one and only one degree  $i$  operator.

The first option gives  $2p_i - 2i - 1 = 0$  which is inconsistent since  $p_i$  and  $i$  are both integer. For the second option, we have  $p_i = i - 1$ . We also need to impose constraints on the numbers  $\delta_{1i}$  and  $\delta_{2i}$ :

$$\delta_{1i} + (i - 1) - 2i + 1 \geq 0, \quad \delta_{2i} + (i - 1) - 2i + 1 \geq 0, \quad (23)$$

This gives

$$(1) : \delta_{1i} \geq i \text{ and } \delta_{2i} \geq i. \quad (24)$$

The case  $\delta_{1i} < i$  and  $\delta_{2i} < i$  is not needed to consider, since this implies the original quiver has negative number of degree  $i$  operator, which is impossible.

We are left with the situation

$$(2) : \delta_{1i} \geq i \text{ and } \delta_{2i} < i; \quad (25)$$

and case

$$(3) : \delta_{1i} < i \text{ and } \delta_{2i} \geq i. \quad (26)$$

For case (2), we can conclude that part 2 does not have a degree  $i$  operator since the maximal contribution of the new appearing puncture to degree  $i$  differential is  $p_i = i - 1$ , and the maximal number of degree  $i$  operator is  $d_i^{(2)} \leq \delta_{2i} + (i - 1) - 2i + 1 < 0$ . Matching the coulomb branch moduli, we have the equation

$$(\delta_{1i} + p_i - 2i + 1) = \delta_{1i} + \delta_{2i} - 2i + 1, \quad (27)$$

or

$$(\delta_{1i} + p_i - 2i + 1) + 1 = \delta_{1i} + \delta_{2i} - 2i + 1, \quad (28)$$

Here  $p_i$  is the contribution to the degree  $i$  operators of the new appearing puncture. We assume the decoupled gauge group does not have a degree  $i$  operator in first option and does have one in second option. Solving the equations, the solution is  $p_i = \delta_{2i}$  or  $p_i = \delta_{2i} - 1$ . The same analysis can be applied to case (3), we have  $p_i = \delta_{1i}$  or  $p_i = \delta_{1i} - 1$ . It seems that we can not uniquely determine the new appearing puncture. We will argue that the second option is not possible.

Let's just consider the case (3), we assume that  $\delta_{1i} = i - a$ , where  $a \geq 1$ . If  $p_i = \delta_{1i} - 1 = i - (a + 1)$ , this means that the  $i$ th box is at the level  $(a + 1) \geq 2$  in the Young tableaux of the new puncture, in this case, the decoupled gauge group has a degree  $i$  operator, so the decoupled gauge group is  $SU(i)$  or  $USp(i)$  ( $USp(i)$  is possible only in the case  $i$  is even). The original quiver is derived by gauging a  $SU(i)$  subgroup of the new appearing puncture. According to our formula (4), the maximal simple subgroup of the new punctures is  $SU(n_1)$ , where  $n_1$  is the number of box of the first row. However, for the new puncture  $n_1 < i$  since  $i$ th box is not in the first row, the maximal simple subgroup of the flavor symmetry is less than  $SU(i)$ . So we conclude the option  $p_i = \delta_{1i} - 1$  is impossible and  $p_i = \delta_{1i}$ .

Combining all the analysis above, we can give a concise formula for  $p_i$

$$p_i = \min(\delta_{1i}, \delta_{2i}, i - 1). \quad (29)$$

and if  $\min(\delta_{1i}, \delta_{2i}) \geq i$ , there is a degree  $i$  operator for the decoupled gauge group. Indeed, in the derivation of the above formula, we did not use the assumption that only two punctures are colliding, this formula is true for colliding any number of punctures. To determine the fundamental fields on the decoupled gauge group, we need to match the dimension of Higgs branch. This can be done in a similar way as we count the Coulomb branch. If the three punctured sphere does not have any Coulomb branch moduli, then this three punctured sphere does not contribute to Higgs branch. If the three punctured sphere is reducible but have non-zero Coulomb branch moduli, say it has a degree  $k < N$  moduli, then this sphere contributes to the Higgs branch. We can not use (8) to calculate the Higgs branch contribution though. To calculate the contribution, we reduce it to a rank  $k$  theory, and reduce the Young tableaux with a total of  $k$  boxes (we simply delete the extra boxes), then we use formula (8) to calculate the contribution to Higgs branch.

We next consider the degeneration of higher genus theory. Let's study Riemann surface with genus  $g$  and  $n$  marked points; there are now three kinds of degeneration: the genus reduces by one after normalization as in Figure 5; or two marked points collide and after normalization we are left a genus  $g$  component and a genus zero component; we also have the degeneration for which we are left with a genus  $g_1$  and genus  $g_2$ , where  $g_1$  and  $g_2$  are nonzero.

For the first case, after normalization, we are left a genus  $g - 1$  surface with  $n + 2$  marked points, and a decoupled gauge group. The Coulomb branch dimension must match, let's assume the new puncture is associated with a nilpotent orbit whose dimension is  $d$ , then we have the following matching condition using formula (7):

$$\frac{1}{2} \sum d_i + \frac{1}{2}(2d) + (g - 1 - 1)(N^2 - 1) + r = \frac{1}{2} \sum d_i + (g - 1)(N^2 - 1) \quad (30)$$

where  $r$  is the rank of the decoupled gauge group and  $d_i$  is the dimension of nilpotent orbit associated with the puncture  $i$ , (The total dimension of Hitchin's moduli space is  $\sum 2d_i + 2(g - 1)(N^2 - 1)$ , there is a Hitchin's fibration and the base has half the dimension of moduli space. The Hitchin's fibration is identified with the Seiberg-Witten fibration, and the dimension of the base is the dimension of the coulomb branch). Solving the above equation, we have

$$d = N^2 - (r + 1). \quad (31)$$

the maximal dimension of  $d$  is the dimension of regular nilpotent orbit and has the dimension  $d = N^2 - N$ , this implies that the minimal value of  $r$  is  $N - 1$ . However, the maximal rank of the decoupled gauge group is  $(N - 1)$ . We conclude that the decoupled gauge group is  $SU(N)$  and the new puncture is a full puncture. We have assumed that the original theory is irreducible and we need to check genus  $(g - 1)$  is also irreducible. The irreducibility of the original theory implies

$$d_g = \frac{1}{2} \sum_i d_i + (g - 1)(N^2 - 1) > 0, \quad (32)$$



This condition is automatically good if  $g \geq 2$ , there is no constraint on the number of punctures. In the case  $g = 1$ , we need to have at least one puncture.

For the genus  $g - 1$  theory, we have the dimension of the Coulomb branch

$$d_{g-1} = \frac{1}{2} \sum_i d_i + N^2 - N + (g-2)(N^2-1) = \frac{1}{2} \sum_i d_i + (g-1)(N^2-1) - (N-1) \quad (33)$$

In the case  $g > 2$ ,  $d_{g-1} > 0$  is always true. In the case of  $g = 1$ , since the minimal dimension of the nilpotent orbit is  $2N - 2$ , we see that  $d_{g-1} \geq 0$ .

One can also confirm our result by matching Higgs branch moduli using (8). The matching condition is

$$\sum_i l_i + 2l + (1 - (g-1))(N-1) - n = \sum_i l_i + (1-g)(N-1), \quad (34)$$

where  $l$  is the contribution of the new puncture and  $n$  is the dimension of the decoupled gauge group, which is  $n = (N^2 - 1)$  in our case. The new puncture is a full puncture and have  $l = \frac{1}{2}(N^2 - N)$ . One can check the above equation is right. This calculation also shows that we do not have any fundamental fields on the  $SU(N)$  gauge group.

The degeneration limit with genus  $g_1$  and  $g_2$  parts can be analyzed similarly. The  $g_1$  component has  $n_1 + 1$  marked points and  $g_2$  component has  $n_2 + 1$  marked points, according to our previous analysis, these two theories are both irreducible. We have the following relation for the coulomb branch dimension

$$\sum k_{1i} + \frac{1}{2}d + (g_1-1)(N^2-1) + \sum k_{2i} + \frac{1}{2}d + (g_2-1)(N^2-1) = \sum (k_{1i} + k_{2i}) + (g_1+g_2-1)(N^2-1) - r \quad (35)$$

where  $r$  is the rank of the decoupled gauge group and  $d$  is the dimension of the nilpotent orbit associated with the puncture as we defined above. Similar analysis shows that the decoupled gauge group is  $SU(N)$  and the new puncture is a full puncture.

The last case with a genus  $g$  component and genus zero component is a little bit different. We know that a genus  $g$  component is irreducible, there are nonzero moduli for each degree. Assume the contribution of two punctures to the moduli of degree  $i$  is  $\delta_{1i}$ , similar analysis with the degeneration limit of genus zero case can be done and we have the following conclusion about the order of poles of the new puncture

$$p_i = \min(\delta_{1i}, i - 1). \quad (36)$$

The decoupled gauge group can be derived by noticing that if  $\delta_{1i} \geq i$ , the decoupled gauge group has a degree  $i$  operator.

Now let's discuss what is the intermediate state in  $A_{N-1}$  conformal Toda field theory side. It is argued in [46], that the primary field corresponding to the simple puncture labeled by the Young tableaux  $[n_1, n_2, n_3, \dots, n_s]$  has the form

$$e^{i\vec{\beta} \cdot \vec{\phi}}, \quad (37)$$

where  $\vec{\phi} = (\phi_1, \dots, \phi_N)$  and  $\sum \phi_i = 0$ , and  $\beta$  has the form

$$\vec{\beta} = \vec{p} - iQ\vec{p}, \quad (38)$$

$\rho$  is a fixed vector and  $\vec{p}$  is a real vector, they are both dictated by the Young tableaux.  $\vec{p}$  has the form:

$$\vec{p} = (\underbrace{p_1, \dots, p_1}_{l_1}, \underbrace{p_2, \dots, p_2}_{l_2}, \dots, \underbrace{p_r, \dots, p_r}_{l_r}), \quad (39)$$

here  $l_i$  is the height of  $i$ th column of the Young tableaux. Notice that in the gauge theory, the mass deformation at the puncture has the same form as (39), so we identify the mass parameters with the momentum  $\vec{p}$  and these numbers are fixed (they are UV parameters).

The intermediate state also has the form (39) and the Young tableaux of it is dictated by the new appearing puncture. The physical momentum of the intermediate state is identified with the Coulomb branch expectation value. For instance, if the new appearing puncture is the full puncture and the decoupled gauge group is  $SU(N)$ , then the intermediate state has the form  $\vec{p} = (a_1, a_2, \dots, a_N)$ ,  $\sum a_i = 0$ , here  $a_i, i \geq 2$  parameterizes the Coulomb branch of  $SU(N)$ . However, in general, not every column of the new appearing puncture can be deformed since not all flavor symmetry of it is gauged. The practical rule is: if  $\min(\delta_{1i}, \delta_{2i}) \geq i$ , the decoupled gauge group has a degree  $i$  operator; since  $p_i = i - 1$  and the  $i$ th box of the new Young tableaux is at the first row, we claim that the  $i$ th column of the new puncture is deformed (we always deform first column so that the traceless condition is satisfied).

## 5 Examples

We present some of the examples in this section to derive the quiver for a specific nodal curve. The six dimensional description of a linear quiver of  $A_N$  type has been worked out by Gaiotto [2], it involves two generic puncture, and several basic punctures.

We decide what happens when a simple puncture is colliding with a generic puncture with rows  $n_1 \geq n_2 \dots \geq n_k$ , and the height of the first column is  $s_1$ . We assume the other part is a degree  $N$  theory, the other cases can be treated similarly.

In this case, the other part has a gauge group  $SU(N)$  and has degree  $i$  moduli, so  $\delta_{2i} \geq i$ . Then the new appearing puncture is determined by  $p_i = \min(\delta_{1i}, i - 1)$ .

We have the following table of the numbers we need

i	2	3	4	...	$n_1$	$n_1 + 1$	...	N
$p_{1i}$	1	1	1	...	1	1	...	1
$p_{2i}$	1	2	3	...	$n_1 - 1$	$n_1 - 1$	...	$N - s_1$
$\delta_{1i}$	2	3	4	...	$n_1$	$n_1$	...	$N - s_1 + 1$
$p_i$	1	2	3	...	$n_1 - 1$	$n_1$	...	$N - (s_1 - 1)$
	1	1	1	...	1	0	...	0

In the last line of table, we indicate whether decoupled gauge group has a degree  $i$  operator. We can see from the table that the decoupled gauge group is  $SU(n_1)$ , and the new puncture has the feature that we combine the first row and second row and leave other rows unchanged. This agrees with the result by Gaiotto. We can now determine the fundamentals on the gauge group  $SU(n_1)$ , since the three punctured sphere does not carry any Coulomb branch moduli and the dimension is negative, there is no contribution to Higgs branch from it. We have

$$\sum_{i=2}^n l_i + l_2 + 1 + (N - 1) = \sum_{i=2}^n l_i + l_2 + n_1 n_2 - (n_1^2 - 1) + (N - 1) + x, \quad (40)$$

Here  $l_2$  is the contribution of the generic puncture, 1 is the contribution of the simple puncture; we have used the fact that the new puncture has the contribution to Higgs branch  $(l_2 + n_1 n_2)$ , where  $x$  is the contribution from the fundamental fields. Calculate it, we get  $x = n_1(n_1 - n_2)$ , so there is  $n_1 - n_2$  fundamentals on  $SU(n_1)$ , this is in agreement with the explicit quiver theory.

Next, let's consider collision of two identical punctures which have two columns with equal height  $N$ , this means that we are considering  $A_{2N-1}$  compactification. We list the analysis in the following table:

i	2	3	4	...	$2k$	$2k + 1$	...	$2N$
$p_{1i}$	1	1	2	...	$k$	$k$	...	$N$
$p_{2i}$	1	1	2	...	$k$	$k$	...	$N$
$\delta_{1i}$	2	2	4	...	$2k$	$2k$	...	$2N$
$p_i$	1	2	3	...	$2k - 1$	$2k$	...	$2N - 1$
	1	0	1	...	1	0	...	1

From the table, we can see that the new puncture is a full puncture and the decoupled gauge group has only even rank operator, the natural decoupled gauge group is  $USp(2N)$ , one may wonder why  $USp$  gauge group appears when we compactify a  $A_{2N-1}$  theory on a Riemann surface, this can be done by including a outer automorphism of the gauge group  $SU(2N)$  in the compactification, see [80, 81]. Let's calculate the fundamentals on  $USP(2N)$ . The three punctured sphere does not contribute to Higgs branch. The two column puncture has Higgs dimension  $N$ , the full puncture contributes  $(2N^2 - N)$ . we have

$$\sum_{i=2}^n l_i + 2N + (N - 1) = \sum_{i=2}^n l_i + (2N^2 - N) + (N - 1) - 2N^2 - N + x, \quad (41)$$

solving this equation, we have  $x = 4N$ , so we have 2 fundamentals on  $USp$  node. This is just what is found by Gaiotto [2], see also [71].

We also confirm another example which is studied in [71]. One puncture has partition  $[2, 2, \dots, 2]$ , the other puncture has partition  $[3, 2, \dots, 2, 1]$ , the data is assembled in following table:

i	2	3	4	...	$2k$	$2k+1$	...	$2N$
$p_{1i}$	1	1	2	...	$k$	$k$	...	$N$
$p_{2i}$	1	2	2	...	$k$	$k+1$	...	$N$
$\delta_{1i}$	2	3	4	...	$2k$	$2k+1$	...	$2N$
$p_i$	1	2	3	...	$2k-1$	$2k$	...	$2N-1$
	1	1	1	...	1	1	...	1

From the table, we conclude that the new puncture is a full puncture and the decoupled gauge group is a  $SU(N)$  gauge group. The three punctured sphere has zero Coulomb branch dimension, its Higgs branch dimension can be calculated using (8), and equals to  $(2N^2 + 2N)$ , we have the equation

$$\sum_{i=2}^n l_i + N + (N+1) + (N-1) = \sum_{i=2}^n l_i + (2N^2 - N) + (N-1) - ((2N)^2 - 1) + x + 2N^2 + 2N, \quad (42)$$

we find  $x = N$ , this sounds weird, since no conventional matter on  $SU(N)$  can give this number for Higgs branch. The three punctured sphere does not represent the conventional matter too, so we may combine the two contribution, the total contribution to Higgs branch of the  $SU(N)$  node is  $2N^2 + 3N = (2N^2 - N) + 4N$ , that's an antisymmetric matter and two fundamentals on it, it splits in this way so that the  $SU(N)$  gauge group is conformal.

Next, let's consider the collision of two generic punctures. First we want to mention a special case when  $\delta_{1N} \geq N - 1$ , this means that  $p_N = N - 1$ , without any calculation, we can conclude that the new puncture is a full puncture, since the maximal value of  $p_N$  is  $N - 1$  and it is possible only if the puncture is a full puncture.

Finally, let's consider an example of collision of two generic punctures, in some cases, the new appearing three punctured sphere has nonzero moduli and is an isolated SCFT. Let's consider collision of two identical punctures with partitions  $[3, 1, 1, 1]$ . The linear quiver gauge theory with these two punctures is depicted in Figure 7a). The six dimensional construction is depicted in Figure 7b). We study another weakly coupled theory corresponding to collide two generic puncture represented by black dot, the nodal curve is depicted in Figure 7c). We write the corresponding generalized quiver corresponding in Figure 7d).

We do need to know what happened when we collide two generic punctures, we follow our method and the data we need is in the following table

i	2	3	4	5	6
$p_{1i}$	1	2	2	2	2
$p_{2i}$	1	2	2	2	2
$\delta_{1i}$	2	4	4	4	4
$p_i$	1	2	3	4	4
	1	1	1	0	0

The new appearing puncture has the partition  $[5, 1]$ , the decoupled gauge group is  $SU(4)$ . The decoupled three punctured sphere is reducible but it carries a degree

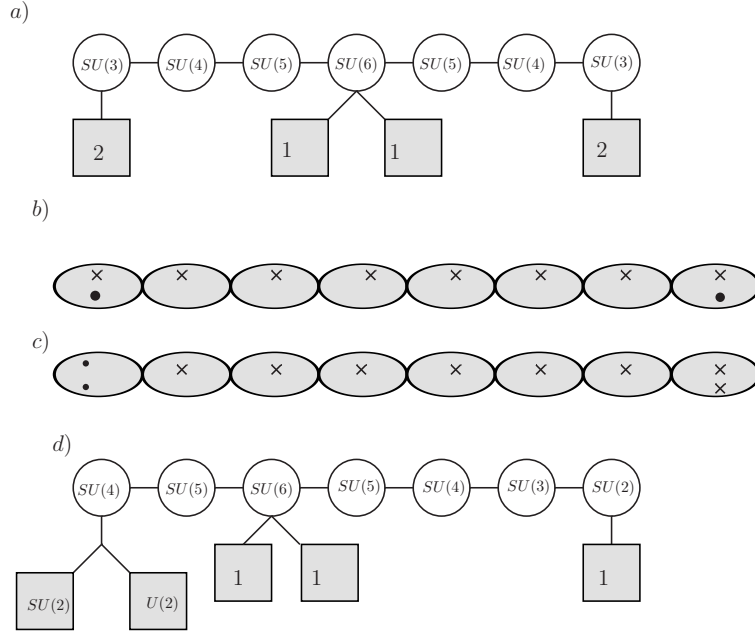


Figure 7: a) A linear quiver, b) The six dimensional construction corresponding to quiver in (a), the cross denotes the simple puncture and the black dot denotes the puncture with partition  $[3, 1, 1, 1]$ ; c) A different weakly coupled gauge group description, we collide two generic punctures; d) A generalized quiver corresponding to (c).

3 moduli as we can check explicitly. To see what this theory is, we reduce the three punctured sphere to a rank 3 theory, then all the three puncture now is the full puncture (we simply delete the extra boxes for each puncture), this theory is the familiar  $E_6$  SCFT. Notice in this case, we use the subgroup  $SU(2) \times U(2) \times SU(4)$  decomposition of  $E_6$  instead of the familiar  $SU(3) \times SU(3) \times SU(3)$  decomposition. After gauging the  $SU(4)$  node, we are left with a  $SU(2) \times U(2)$  flavor symmetry. Combining the  $U(1)$  flavor symmetry on  $SU(6)$  node, we have  $U(2) \times U(2)$  flavor symmetry on this quiver tail, and these are represented by two generic punctures with partition  $[3, 1, 1, 1]$ . To confirm our identification of  $E_6$  theory, we can match the Higgs branch dimension. According to our rule, the contribution of three punctured sphere should be calculated using rank 3 theory, and its Higgs branch has dimension 11. The total dimension of Higgs branch of quiver depicted in Figure 7a) is 19 using our formula (8). The Higgs branch dimension of the quiver in Figure 7d) after the complete degeneration is

$$23 + 11 + x - 15 = 19, \quad (43)$$

where 23 is from the left quiver and  $x$  is the contribution from the fundamental fields, we have  $x = 0$ . This result shows that there is no extra fundamentals on  $SU(4)$  node, which means that  $E_6$  matter system provides conformal anomaly like the three fundamentals on  $SU(4)$  node, one can check this using the method in [1].

## 6 Three punctured sphere and Sewing SCFT

As we discussed in section III, the punctured Riemann surface can be derived by gluing the three punctured spheres. In gauge theory terms, we may think that each three puncture corresponds to a matter system, say a bi-fundamental field or a strongly coupled SCFT like  $E_6$  theory [79], and the whole generalized quiver gauge theory is derived by gauging various flavor symmetry of the matter system represented by the three punctured sphere. This is not true in general as we explain now. Six dimensional construction indicates that each gauge group corresponds to a node of the nodal curve, and the node connects to only two three punctured spheres. However, for the  $A_N$  linear quiver, there are indeed two bi-fundamental fields for the middle gauge group, but we also have the fundamental fields to make the theory conformal, so the gauge theory can not be thought as naively gluing different matter systems. The virtue of six dimensional construction is to somehow combine the description of some of the fundamental fields into a single puncture and this makes it possible that the four dimensional gauge theory can be thought of as gluing three punctured sphere, but not every three punctured sphere represents a real four dimensional matter system.

However, in some cases, the gauge theory can indeed be derived by gluing the matter system. This is true for any generalized  $SU(2)$  quiver gauge theory, since this theory has only one type puncture and we have only one type three punctured sphere, this represents the bi-fundamental fields with flavor symmetry  $SU(2) \times SU(2)$ , any generalized  $SU(2)$  quiver gauge theory in any duality frame can

be derived by gluing the flavor symmetry of the basic three punctured sphere. Another interesting example is  $\underbrace{SU(N) - SU(N) - \dots - SU(N)}_n$  gauge theory with  $N$

fundamentals at each end, this theory is described as six dimensional  $A_{N-1}$  theory compactified on a sphere with  $n + 1$  simple punctures and two full punctures. At one duality frame corresponding to the conventional description, the six dimensional nodal curve consists of three punctured sphere with two full punctures and one simple puncture, each three punctured sphere represents the bi-fundamental fields between two  $SU(N)$  gauge groups. It is obvious that the gauge theory is indeed derived by gluing the bi-fundamental fields represented by the three punctured sphere. Another example is the theory derived by compactifying six dimensional theory on a genus  $g$  Riemann surface, as we discussed in previous section, in the complete degeneration limit, all three punctured sphere has three full punctures and this is the so-called  $T_N$  theory, see [2, 82].

To see the case that not every three punctured sphere represents a matter system, Let's look at the conformal quiver  $SU(3) - SU(2)$ , this quiver is very simple but it is good enough for our purpose. The quiver and the degeneration limit for one weakly coupled limit is shown in Figure 8. In Figure 8d), we draw the complete degeneration limit, the  $SU(2)$  gauge group couples to two three punctured sphere, sphere 1 with two full punctures and one simple puncture and sphere 2 with two simple punctures and one full puncture. From the quiver gauge theory, we see that  $SU(2)$  gauge theory couples to a bi-fundamental fields between two  $SU(3)$  gauge group and one fundamental fields to itself. The sphere 1 represents bi-fundamental fields between two  $SU(3)$  gauge group, we may think sphere 2 represents the fundamental field on  $SU(2)$ , this is obviously not the case, the flavor symmetry of the fundamental on  $SU(2)$  is only  $SU(2) \times U(1)$  while the sphere 2 has flavor symmetry  $SU(3) \times U(1) \times U(1)$ .

We now conjecture that a three punctured sphere represents a real matter system if the three punctured sphere is irreducible or the moduli space is just a point. The  $E_6$  SCFT theory is irreducible while the bi-fundamental fields between two  $SU(N)$  gauge group has zero-dimensional moduli space.

Let's now discuss the interesting relation with conformal field theory. For Liouville theory, the correlation function is decomposed as the product of the three point functions and the conformal block [83]. We now find an interpretation of this fact in terms of  $N = 2$  gauge theory. For  $N = 2$   $SU(2)$  generalized quiver gauge theory, there is only one three punctured sphere, and this represents a real matter system. In any duality frame, we can think of the gauge theory as derived by gauging the flavor symmetry of the real matter system represented by the three punctured sphere. This property of gauge theory dictates that the Liouville theory correlation function can be written as the product of three point functions and the conformal block. In the conformal Toda theory, it is shown that generically the correlation function can not be written as the product of three point function and the conformal block [11, 84, 85], however, in some special cases the correlation function is factorizable. There is one

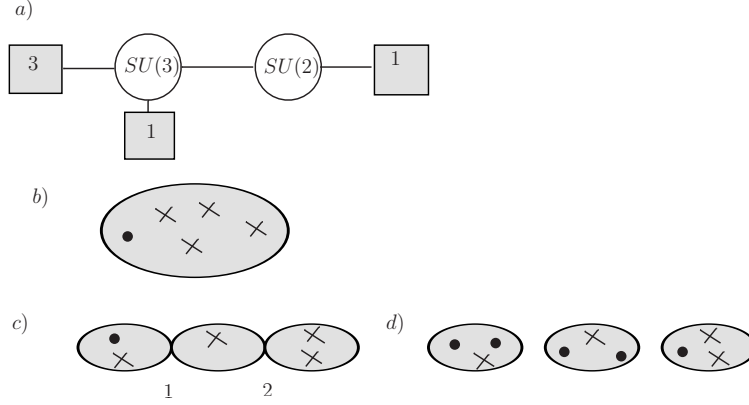


Figure 8: a) A conformal quiver with gauge group  $SU(3) \times SU(2)$ ; b) The six dimensional construction involves a Riemann surface with 5 punctures, the cross means the simple puncture and the black dot represents full puncture with flavor symmetry  $SU(3)$ ; c) The degeneration limit corresponding to the weakly coupled quiver in a, node 1 denotes  $SU(3)$  gauge group and node 2 denotes  $SU(2)$  gauge group; d) The complete degeneration limit, new punctures appear.

example shown in [11], the gauge theory is  $\underbrace{SU(N) - SU(N) - \dots - SU(N)}_n$  we considered previously, and all the three punctured spheres in this weakly coupled duality frame represent real matter system. Notice that in other duality frame, some of the three punctured sphere does not represent a real matter system and the correlation function in that channel is not factorizable. We then have this conjecture:

The correlation function in one channel of two dimensional CFT is factorizable if all the three punctured sphere for the corresponding weakly coupled gauge description represents real matter system.

Since each channel of 2d CFT correlation function corresponds to a duality frame of the gauge theory, this observation makes a completely determination about whether the correlation function can be written as the product of the three point functions and the conformal block.

We now prove a theorem that the irreducible three punctured sphere (represents a real matter system) must have a full puncture if it is derived from an degeneration limit of an irreducible rank  $N$  theory and  $\delta_{1N} \leq \delta_{2N}$ . Since we have a non-zero degree  $N$  moduli, this implies that  $p_N + \delta_{1N} \geq 2N$ . First, let's assume  $\delta_{1N} \leq \delta_{2N}$ , since  $p_N = \min(\delta_{1N}, N-1)$ , if  $p_N = \delta_{1N}$ , we have  $p_N + \delta_{1N} < N-1 + N-1 = 2N-2 < 2N$  (we have used the fact that the maximal value of  $p_N$  is  $(N-1)$ ), so we must have  $p_N = N-1$ , this implies that the new puncture is a full puncture.



## 7 Conclusion

In this paper, we studied four dimensional  $N = 2$  generalized quiver gauge theory. These theories are derived from six dimensional  $(0, 2)$  SCFT compactified on a Riemann surface with some punctures. We argue that different duality frames correspond to different nodal curves of the same punctured Riemann surface. We give an explicit formula to determine the weakly coupled gauge group. We can also determine the matter only ending on the decoupled gauge group. With this result, we can determine the weakly coupled gauge group description in any duality frame for a given generalized quiver gauge theory. This information also tells us what is the intermediate state of two dimensional conformal field theory which is important for AGT duality. The same analysis can be carried to the asymptotical free theory.

The gauge theory calculation is amazingly simple in determining the intermediate state. It is interesting to calculate the intermediate state directly from conformal Toda field theory using operator product expansion, this may confirm our calculation. The method we presented in this paper may give some hints. We also give a conjecture that whether the 2d correlation function can be written as the product of three point functions and the conformal block depends on the three punctured sphere of the complete degeneration limit. It is interesting to check this conjecture using conformal field theory technic.

We believe that the method we present here can also be applied to four dimensional  $N = 2$   $A$  type quiver with  $USp - SO$  group [68, 86, 87].

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## A The Contribution to Higgs Branch of One Generic Puncture

The Higgs branch of the generalized quiver is a little bit difficult. For simplicity, we first study the Riemann sphere two generic punctures and  $s_1 + s_2$  simple punctures, we have the explicit lagrangian description in one dual frame. Let's first consider the contribution of the simple puncture. Adding one simple puncture, in the corresponding quiver, we add a  $SU(N)$  node to the original quiver, we need to add a bifundamental matter between two  $SU(N)$  nodes, so the net contribution to the Higgs branch is  $N^2 - (N^2 - 1) = 1$ .

Now consider a generic puncture labeled by  $[n_1, n_2, \dots, n_s]$ , we have described the quiver tail to this puncture in the main part of the paper, it is  $SU(k_1) - SU(k_2) - \dots - SU(N)$ , where  $k_j = \sum_{i=1}^j n_i$ . For present purpose, we split the  $SU(N)$  gauge

group into two equal parts, the rank of the gauge group for this tail is  $\frac{1}{2}(N^2 - 1)$ , the fundamental contribution to  $SU(N)$  from this tail is  $N - k_{s-1}$ .

The contribution from bifundamental matter of this quiver tail is

$$\sum_{j=1}^{s-1} k_j k_{j+1}. \quad (44)$$

The contribution from the fundamental matter is

$$\sum_{j=1}^{s-1} (2k_j - k_{j-1} - k_{j+1})k_j + N(N - k_{s-1}), \quad (45)$$

where  $k_j$  is the rank of  $j$ th gauge group and we set  $k_0 = 0, k_m = N$ ,  $k_j$  is related to the Young tableaux as  $k_j = n_1 + \dots n_j$ . The Higgs branch of this tail is (we assume the gauge group is completely higgsed in Higgs branch)

$$\sum_{j=1}^{s-1} k_j k_{j+1} + \sum_{j=1}^{s-1} (2k_j - k_{j-1} - k_{j+1})k_j + N(N - k_{s-1}) - \sum_{j=1}^{s-1} (k_j^2 - 1) - \frac{1}{2}(N^2 - 1). \quad (46)$$

After some calculation, we have

$$\sum_{j=1}^s k_j (k_j - k_{j-1}) + s - 1 - \frac{1}{2}(N^2 - 1) = \frac{1}{2}(\sum_{i=1}^s n_i^2 - N) + s + \frac{N - 1}{2}. \quad (47)$$

The first term in above form is thought to be the contribution to Higgs branch due to this generic puncture. Since for this tail, we need  $s$  simple punctures, and the contribution of the simple puncture to Higgs branch is 1, so the second term is thought of the contribution from simple punctures. The last term is thought to be the global contribution just like  $-(N^2 - 1)$  term for the Coulomb branch. Finally, adding two tails, there are a  $(N-1)$  extra terms, which we think as the total global contribution on the sphere. The total dimension for the Higgs branch is then

$$\sum_i l_i + N - 1, \quad (48)$$

where

$$l_i = \frac{1}{2}(\sum_{j=1}^s n_j^2 - N). \quad (49)$$

For other generalized quiver, since we do not have a lagrangian description, we do not know how to count the dimension of Higgs branch. However, based on the analysis for the linear quiver, we conjecture that for each puncture, the contribution to the Higgs branch is

$$l_k = \frac{1}{2}(\sum_{i=1}^s n_i^2 - N), \quad (50)$$

and there is a  $N - 1$  global contribution to Higgs branch. The total dimension of Higgs branch is

$$\sum_i l_i + (N - 1). \quad (51)$$

The generalization to higher loop case is straightforward

$$\sum_i l_i + (1 - g)(N - 1). \quad (52)$$

In fact, for the higher loop case, we really count the difference between the dimension of the matter and the dimension of the gauge group, and in some cases, the theory is not completely higgsed, there are some unbroken  $U(1)$  gauge symmetry.

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